q-series

in

Number Theory

and

Combinatorics

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Abstract

Srinivasa Ramanujan (1887-1920) was one of the world's greatest mathematical geniuses. He work extensively in a branch of mathematics called "q-series". Around 1913, he found an important formula which now is known as Ramanujan's ${}_1\psi_1$ summation formula.

The aim of this thesis is to investigate Ramanujan's ${}_1\psi_1$ summation formula and explore its applications to number theory and combinatorics. First, we consider several classical important results on elliptic functions and then give new proofs of these results using Ramanujan's ${}_1\psi_1$ summation formula. For example, we will present a number of classical and new solutions for the problem of representing an integer as sums of squares (one of the most celebrated in number theory and combinatorics) in this thesis. This will be done by using q-series and Ramanujan's ${}_1\psi_1$ summation formula. This in turn will give an insight into how Ramanujan may have proven many of his results, since his own proofs are often unknown, thereby increasing and deepening our understanding of Ramanujan's work.

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Contents