Ramanujan-type series for $\frac{1}{\pi}$ with quadratic irrationals

A thesis presented in partial fulfillment
of the requirements for the degree of

Master of Science
in
Mathematics

at Massey University, Albany,
New Zealand.

Antesar Mohammed Aldawoud

2012

Abstract

In 1914, Ramanujan discovered 17 series for $1/\pi$, 16 are rational and one is irrational. They are classified into four groups depending on a variable ℓ called the level, where $\ell = 1$, 2, 3 and 4. Since then, a total of 36 rational series have been found for these levels. In addition, 57 series have been found for other levels. Moreover, 14 irrational series for $1/\pi$ were found. This thesis will classify the series that involve quadratic irrationals for the levels $\ell \in \{1, 2, 3, 4\}$. A total of 90 series are given, 76 of which are believed to be new. These series were discovered by numerical experimentations using the mathematical software tool "Maple" and they will be listed in tables at the end of this thesis.

Acknowledgements

I am heartily thankful to my supervisor Dr. Shaun Cooper, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject. I am grateful to all my family and friends, most importantly to my husband Saud, whose love and guidance is with me in whatever I pursue. Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the project.

Contents

List of Tables

Chapter 1

Introduction

In 1914, Ramanujan [15] gave 17 extraordinary series for $1/\pi$. Two of his best known examples are

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\ \frac{1}{396^{4n}}\,(1103+26390n), \tag{1.1}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty}\frac{(2n)!^3}{n!^6}\ \frac{1}{2^{12n}}\,(42n+5). \tag{1.2}$$

These series have excellent convergence properties. They are relatively scarce, and the numbers and the coefficients are striking. They are widely famous, nearly every mathematician will recognize (1.1). The proofs of these series require sophisticated number theory, such as modular equations and class invariants. The series (1.1) converges very fast: each term contributes 8 digits of π.

The ratio test for the series (1.1) shows: if

$$t_n = \frac{(4n)!}{(n!)^4}\ \frac{(1103+26390n)}{396^{4n}},$$

Then :

$$\begin{aligned}\frac{t_{n+1}}{t_n} =& \frac{(4n+4)!}{((n+1)!)^4}\ \frac{(1103+26390(n+1))}{396^{4n+4}}\cdot\frac{(n!)^4}{(4n)!}\ \frac{396^{4n}}{(1103+26390n)},\\ =& \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4}\frac{1}{396^4}\frac{(1103+26390(n+1))}{(1103+26390n)}.\end{aligned}$$

Taking the limit:

$$\begin{aligned}\lim_{n\to\infty}\frac{t_{n+1}}{t_n} &= \lim_{n\to\infty}\frac{4^4}{396^4},\\ &= \frac{1}{99^4},\\ &\simeq 10^{-8}.\end{aligned}$$

In 1985, Gosper programmed (1.1) on a computer and got a world record of more than 17 million digits of π [14].

Baruah, Berndt and Chan (2009) wrote in the survey [1]: "The series (1.2) appeared in the "Walt Disney" film *High School Musical*, starring Vanessa Anne Hudgens, who plays an exceptionally bright high school student named Gabriella Montez. Gabriella points out to her teacher that she had incorrectly written the left-hand side of (1.2) as $8/\pi$ instead of $16/\pi$ on the blackboard, After first claiming that Gabriella is wrong, her teacher checks (possibly *Ramanujan's Collected Papers?*) and admits that Gabriella is correct. Formula (1.2) was correctly recorded on the blackboard" (p. 568).

In the paper [6], the authors derived a general statement that encapsulates all known rational analogues of Ramanujan's series for $1/\pi$.

In this thesis, I will classify new series for $1/\pi$ that involve quadratic irrationals. An example of one of our series for $1/\pi$ from level 3 is given by:

$$\frac{1}{\pi} = 2\sqrt{1-108x}\sum_{k=0}^{\infty}\frac{(3k)!(2k)!}{(k!)^5}\,(k+\lambda)x^k,$$

with $x = -\frac{1}{6} + \frac{7}{72}\sqrt{3}$ and $\lambda = \frac{5}{22} - \frac{1}{22}\sqrt{3}$.

I include a list of all the known quadratic irrational series for $1/\pi$. Of the 90 series listed in Tables 3.1–3.10, 14 were known previously and 76 are believed to be new.

1.1 Literature Review

In 1914, Ramanujan published a paper entitled"Modular equations and approximations to π" in England [15]. In the paper, he gave 17 series for $1/\pi$. The first three are (1.2) and :

$$\frac{4}{\pi} = \sum_{n=0}^{\infty}\frac{(2n)!^3}{n!^6}\,\frac{1}{2^{8n}}\,(6n+1), \tag{1.3}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty}\frac{(2n)!^3}{n!^6}\left(\frac{3-\sqrt{5}}{16}\right)^{4n}\left((42\sqrt{5}+30)n+(5\sqrt{5}-1)\right). \tag{1.4}$$

He classified the 17 series into four types based on four functions:

$$q_1 = \exp\left(-2\pi\,\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right), \qquad q_2 = \exp\left(-\frac{2\pi}{\sqrt{2}}\,\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right)$$

$$q_3 = \exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right), \qquad q_4 = \exp\left(-\pi\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)$$

where ${}_2F_1$ is the hypergeometric function defined by:

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n. $$

All of Ramanujan's examples above have the property $a+b=c=1$. The function q_4 is the classical theory of Jacobi. While, q_1, q_2 and q_3 are new theories of Ramanujan. Of the 17 series, two are based on q_1, ten are based on q_2, two based on q_3 and the remaining three series are given by (1.2)–(1.4) above, are based on q_4.

We define a series as a "rational" series for $1/\pi$, if C/π can be expressed as a series of rational numbers for some algebraic number C.

The main contributions to Ramanujan series for $1/\pi$

Many books and survey papers have been written to discuss Ramanujan's prominent work. One example is the valuable survey done by Nayandeep D. Baruah, Bruce C. Berndt and Heng Huat Chan [1].

Here is a summary of some of the main contributions to Ramanujan's series for $1/\pi$, and the information in this section comes mainly from [6] and [1].

Fourteen years after Ramanujan's work, Sarvadaman Chowla [8] proved the series (1.3) of Ramanujan's series for $1/\pi$. In 1985, R. William Gosper programmed Ramanujan's series (1.1) which is based on q_2 on a computer. He calculated 17,526,100 digits of π which was at that time a world record. The problem with Gosper's calculation was the series had not been proved yet [1]. In 1987, David and Gregory Chudnovsky [9] developed a theory and derived new series representations for $1/\pi$ and used one of them, the following series which is based on q_1:

$$ \frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3(3n)!} \frac{1}{(640320)^{3n+3/2}} (13591409 + 545140134n), $$

to calculate 2,260,331,336 digits of π which was also a world record in 1989. The above series yields 14 digits of π per term and is the fastest convergent rational series for $1/\pi$.

In 1987, Jonathan and Peter Borwein [3] proved all 17 of Ramanujan's series for $1/\pi$ successfully, they were the first to give complete proofs. In [4] they listed all the series for $1/\pi$ which are based on the function q_1 that were discovered by the Chudnovskys. They also gave one that is new. In 2001, Chan, Wen-Chin Liaw and Victor Tan [7] found new identities helped them to prove the following series:

$$\frac{1}{\pi} = \frac{1}{1500\sqrt{3}} \sum_{n=0}^{\infty} \frac{(3n)!(2n)!}{(n!)^5} \frac{(-1)^n}{300^{3n}} \ (14151n + 827).$$

This series had not been discovered by Ramanujan and it corresponds to q_3. In 2001, Berndt and Chan [2] determined a series for $1/\pi$ that corresponds to q_1 that is not rational which yields about 73 or 74 digits per term that appears to be one of the fastest known convergent series for $1/\pi$. In 2012, Chan and Shaun Cooper [6] introduced classification by level and stated a general theorem that is satisfied by 93 rational series for $1/\pi$, 40 of which were discovered by them. Also Cooper in [11], derived three new series for $1/\pi$. Some mathematicians, namely Takeshi Sato and Matthew Rogers found series for $1/\pi$ based on different functions, these series can be found in [6], [16] and [17].

1.2 Series for $1/\pi$ that involve quadratic irrationals

A series is a "quadratic-irrational" series for $1/\pi$, if C/π can be expressed as a series of quadratic irrational numbers for some algebraic number C. The first non-rational series for $1/\pi$ is (1.4), it was given by Ramanujan in [15]. Also, the Borweins gave 14 non-rational series that involve quadratic irrationals which correspond to q_1 and q_2 [4]. These series are listed in Tables 3.1–3.5 denoted by "*".

Chapter 2

Background theory

2.1 Preliminary Results

In this section we present the definitions that will be used to find the series for $1/\pi$. Let q be a complex number that satisfies $|q| < 1$.

Ramanujan's Eisenstein series and Dedekind's eta function

Ramanujan's Eisenstein series are defined by:

$$P = P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \tag{2.1}$$

$$Q = Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j}, \tag{2.2}$$

$$R = R(q) = 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q^j}{1-q^j}. \tag{2.3}$$

Let

$$\eta_n = \eta_n(q) = q^{n/24} \prod_{j=1}^{\infty} (1 - q^{nj}). \tag{2.4}$$

Another formula for the eta function given by Euler is [3]:

$$\eta_n = \eta_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j \; q^{(6j-1)^2/24}. \tag{2.5}$$

This function, when $n = 1$, is Dedekind's eta-function. The eta functions that will be encountered in this thesis are:

$$\left(\frac{\eta_2}{\eta_1}\right)^{24}, \quad \left(\frac{\eta_3}{\eta_1}\right)^{12} \text{ and } \left(\frac{\eta_4}{\eta_1}\right)^{8}.$$

All these functions have leading term q, so there will be no ambiguity in the branch of $q^{n/24}$ in (2.4).

2.2 The Level

The function x defined in Theorem 1 in the next section satisfies an involution of the form:

$$x\left(e^{-2\pi\sqrt{t/\ell}}\right) = x\left(e^{-2\pi/\sqrt{t\ell}}\right), \qquad t > 0.$$

In this thesis, the positive integer ℓ will be called the level. As stated before, Ramanujan's series are based on the four functions q_ℓ, $\ell \in \{1,2,3,4\}$. For reasons to use the level see [10].

2.3 The Main Result

In this section, we introduce the next result for each level which gives 90 quadratic-irrational series for $1/\pi$ of 4 different types given by (2.7).
The results in this section are from [6].

Theorem 2.3.1 *Let $\ell \in \{1,2,3,4\}$. Let $\omega = \omega(q)$ and $(a,b) \in \mathbb{Z}^2$ be as in Table 2.1. Let $s(k)$ be the sequence defined by the recurrence relation:*

$$(k+1)^2 s(k+1) = (ak^2 + ak + b)s(k)$$

and initial conditions

$$s(-1) = 0, \qquad s(0) = 1.$$

Let

$$x = x(q) = \omega(1 - a\omega). \tag{2.6}$$

Let N be a positive integer called the degree. Either: let ρ and q take the values:

$$\rho = 2\pi\sqrt{N/\ell} \quad \text{and} \quad q = \exp(-\rho);$$

Or

$$\rho = \begin{cases} 2\pi\sqrt{N/4\ell} & \text{if } \ell \equiv 1 \pmod 2, \\ 2\pi\sqrt{N/2\ell} & \text{if } \ell \equiv 2 \pmod 4, \\ 2\pi\sqrt{N/\ell} & \text{if } \ell \equiv 0 \pmod 4 \end{cases} \quad \text{and} \quad q = -\exp(-\rho).$$

Then the identity:

$$\sqrt{1-4ax}\sum_{k=0}^{\infty}\binom{2k}{k}s(k)(k+\lambda)x^k=\frac{1}{\rho} \tag{2.7}$$

holds, provided the series converges.

In Tables 3.1–3.10 we give 90 sets of values of ℓ, N, x and λ. All these values form series for $1/\pi$ that involve quadratic irrationals.
The parameter λ is given by the following result of Chan, Chan and Liu [5]. First we need the following definition [10]:

Definition 2.3.2 *For* $1 \le \ell \le 4$, *define* $z_\ell = z_\ell(q)$ *by*

$$z_\ell = z_\ell(q) = \begin{cases} (Q(q))^{1/4} & \text{if } \ell = 1, \\ \left(\frac{\ell P(q^\ell)-P(q)}{\ell-1}\right)^{1/2} & \text{if } \ell = 2, 3 \text{ or } 4. \end{cases}$$

Theorem 2.3.3 *Let* Z, x, u *and* $h(k)$ *be defined by:*

$$Z = Z(q) = z_\ell^2,$$

$$x = x(q) = \omega(1-a\omega),$$

$$u = u(q) = \sqrt{1-4ax}$$

and

$$h(k) = \binom{2k}{k}s(k).$$

Suppose $t > 0$. *Suppose* $x = x(q), Z = Z(q)$ *and* $u = u(q)$ *satisfy the properties*

$$tZ\left(\exp -2\pi\sqrt{t\ell}\right) = Z\left(\exp -2\pi\sqrt{t/\ell}\right),$$

$$Z(q) = \sum_{k=0}^{\infty} = h(k)x^k(q),$$

and

$$q\frac{d}{dq}\log x(q) = u(q)Z(q).$$

For any integer $N \ge 2$, *let*

$$M(q) = \frac{Z(q)}{Z(q^N)}.$$

Let λ, X *and* U *be defined by*

$$\lambda = \frac{x}{2N}\left.\frac{dM}{dx}\right|_{q=e^{-2\pi/\sqrt{N\ell}}},$$

$$X = x(e^{-2\pi\sqrt{N/\ell}}),$$

$$U = u(e^{-2\pi\sqrt{N/\ell}}).$$

Then

$$\sqrt{\frac{\ell}{N}}\frac{1}{2\pi} = U\sum_{k=0}^{\infty} h(k)(k+\lambda)X^k.$$

The identity (2.7) can be used to prove Ramanujan's series for $1/\pi$, provided that x is defined by (2.6) can be evaluated for specific values of q for various N. The Table 2.1 contains the definitions of the modular forms in terms of the results 2.1–2.4, for each level as well as the recurrence relations and their solutions in terms of the binomial coefficients.

ℓ	(a,b)	$\omega(q)$	$s(k)$	$z_\ell = z_\ell(q)$	$z_\ell = z_\ell(x)$
1	(432,60)	$\frac{1}{864}\left(1-\frac{R(q)}{Q(q)^{3/2}}\right)$	$\binom{6k}{3k}\binom{3k}{k}$	$(Q(q))^{1/4}$	${}_2F_1\left(\frac{1}{6},\frac{5}{6};1;x\right)$
2	(64,12)	$\frac{\eta_2^{24}}{\eta_1^{24}+64\eta_2^{24}}$	$\binom{4k}{2k}\binom{2k}{k}$	$(2P(q^2)-P(q))^{1/2}$	${}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x\right)$
3	(27,6)	$\frac{\eta_3^{12}}{\eta_1^{12}+27\eta_3^{12}}$	$\binom{3k}{k}\binom{2k}{k}$	$\left(\frac{3P(q^3)-P(q)}{2}\right)^{1/2}$	${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)$
4	(16,4)	$\frac{\eta_4^{8}}{\eta_1^{8}+16\eta_4^{8}}$	$\binom{2k}{k}^2$	$\left(\frac{4P(q^4)-P(q)}{3}\right)^{1/2}$	${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$

Table 2.1: for the level

Here is a summary for the results and definitions for each level:

2.3.1 Level 1

Let z_1 be defined as in Table 2.1;

$$z_1 = z_1(q) = (Q(q))^{1/4},$$

then the branch of the root is determined by requiring $z_1 = 1$ when $q = 0$.

Define the modular function ω to be:

$$\omega = \omega(q) = \frac{1}{864}\left(1 - \frac{R(q)}{Q(q)^{3/2}}\right).$$

By substituting the values of $a = 432$ and ω into (2.6) we define x for level 1 to be:

$$x = x(q) = \omega(1 - 432\omega) = \frac{1}{1728}\left(\frac{Q(q)^3 - R(q)^2}{Q(q)^3}\right).$$

Theorem 4.8 in [10] gives:

$$z_1 = z_1(x) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; x\right).$$

The series (2.7) for $\ell = 1$ becomes:

$$\sqrt{1-1728x}\sum_{k=0}^{\infty} \frac{(6k)!}{(k!)^3(3k)!}\,(k+\lambda)x^k = \frac{\sqrt{1/N}}{2\pi}, \tag{2.8}$$

when $q = \exp\left(-2\pi\sqrt{N}\right)$. And when $q = -\exp\left(-\pi\sqrt{N}\right)$ it becomes:

$$\sqrt{1-1728x}\sum_{k=0}^{\infty} \frac{(6k)!}{(k!)^3(3k)!}\,(k+\lambda)x^k = \frac{\sqrt{1/N}}{\pi}. \tag{2.9}$$

Both series (2.8) and (2.9) converge for $|x| < \frac{1}{1728}$. The series (2.8) holds for 7 values of x and λ given in Table 3.1, while there are 4 series for $1/\pi$ satisfied by the series (2.9) given in Table 3.2.

2.3.2 Level 2

Let z_2 be defined as in Table 2.1;

$$z_2 = z_2(q) = (2P(q^2) - P(q))^{1/2},$$

then the branch of the root is determined by requiring $z_2 = 1$ when $q = 0$.

Define the modular function ω to be:

$$\omega = \omega(q) = \frac{\eta_2^{24}}{\eta_1^{24} + 64\eta_2^{24}}.$$

By substituting the values of $a = 64$ and ω into (2.6) we define x for level 2 to be:

$$x = x(q) = \omega(1 - 64\omega) = \frac{\eta_1^{24}\eta_2^{24}}{(\eta_1^{24} + 64\eta_2^{24})^2}.$$

Theorem 4.8 in [10] gives:

$$z_2 = z_2(x) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right).$$

The series (2.7) for $\ell = 2$ becomes:

$$\sqrt{1 - 256x}\sum_{k=0}^{\infty} \frac{(4k)!}{(k!)^4}\,(k + \lambda)x^k = \frac{\sqrt{2/N}}{2\pi}, \tag{2.10}$$

when $q = \exp\left(-2\pi\sqrt{N/2}\right)$. And when $q = -\exp\left(-\pi\sqrt{N}\right)$, it becomes:

$$\sqrt{1 - 256x}\sum_{k=0}^{\infty} \frac{(4k)!}{(k!)^4}\,(k + \lambda)x^k = \frac{\sqrt{1/N}}{\pi}. \tag{2.11}$$

Both series (2.10) and (2.11) converge for $|x| < \frac{1}{256}$. The series (2.10) holds for 9 values of x and λ given in Table 3.3 and Table 3.4, while there are 7 series for $1/\pi$ satisfied by (2.11) given in Table 3.5.

2.3.3 Level 3

Let z_3 be defined as in Table 2.1;

$$z_3 = z_3(q) = \left(\frac{3P(q^3) - P(q)}{2}\right)^{1/2},$$

then the branch of the root is determined by requiring $z_3 = 1$ when $q = 0$.

Define the modular function ω to be:

$$\omega = \omega(q) = \frac{\eta_3^{12}}{\eta_1^{12} + 27\eta_3^{12}}.$$

By substituting the values of $a = 27$ and ω into (2.6) we define x for level 3 to be:

$$x = x(q) = \omega(1 - 27\omega) = \frac{\eta_1^{12}\eta_3^{12}}{(\eta_1^{12} + 27\eta_3^{12})^2}.$$

Theorem 4.8 in [10] gives:

$$z_3 = z_3(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right).$$

The series (2.7) for $\ell = 3$ becomes:

$$\sqrt{1 - 108x}\sum_{k=0}^{\infty} \frac{(3k)!(2k)!}{(k!)^5}\,(k + \lambda)x^k = \frac{\sqrt{3/N}}{2\pi}, \tag{2.12}$$

when $q = \exp\left(-2\pi\sqrt{N/3}\right)$. And when $q = -\exp\left(-\pi\sqrt{N/3}\right)$, it becomes:

$$\sqrt{1 - 108x}\sum_{k=0}^{\infty} \frac{(3k)!(2k)!}{(k!)^5}\,(k + \lambda)x^k = \frac{\sqrt{3/N}}{\pi}. \tag{2.13}$$

Both series (2.12) and (2.13) converge for $|x| < \frac{1}{108}$. The series (2.12) holds for 12 values of x and λ given in Table 3.6, while there are 20 series for $1/\pi$ satisfied by (2.13) given in Tables 3.7 and 3.8.

2.3.4 Level 4

Let z_4 be defined as in Table 2.1;

$$z_4 = z_4(q) = \left(\frac{4P(q^4) - P(q)}{3}\right)^{1/2},$$

then the branch of the root is determined by requiring $z_4 = 1$ when $q = 0$. Define the modular function ω to be:

$$\omega = \omega(q) = \frac{\eta_4^8}{\eta_1^8 + 16\eta_4^8}.$$

By substituting the values of $a = 16$ and ω into (2.6) we define x for level 4 to be:

$$x = x(q) = \omega(1 - 16\omega) = \frac{\eta_1^8\eta_4^8}{(\eta_1^8 + 16\eta_4^8)^2}. \tag{2.14}$$

Theorem 4.8 in [10] gives:

$$z_4 = z_4(x) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

The series (2.7) for $\ell = 4$ becomes:

$$\sqrt{1 - 64x}\sum_{k=0}^{\infty} \frac{(2k)!^3}{(k!)^6}\,(k + \lambda)x^k = \frac{\sqrt{1/N}}{\pi}, \tag{2.15}$$

when $q = \exp(-\pi\sqrt{N})$. And when $q = -\exp(-\pi\sqrt{N})$, it becomes:

$$\sqrt{1-64x}\sum_{k=0}^{\infty}\frac{(2k)!^3}{(k!)^6}\,(k+\lambda)x^k = \frac{\sqrt{1/N}}{\pi}. \tag{2.16}$$

Both series (2.15) and (2.16) converge for $|x| < \frac{1}{64}$. The series (2.15) holds for 8 values of x and λ given in Table 3.9, while there are 9 series for $1/\pi$ satisfied by (2.16) given in Table 3.10.Series for $1/\pi$ of other levels are discussed in the conclusion.

Chapter 3

Methodology: Maple

In our search for series for $1/\pi$, we performed numerical experiments using Maple to compute the values of x and λ. For both cases $q < 0$ and $q > 0$, and for $\ell = 1, 2, 3, 4$, we examined every degree up to $N = 1000$. Let x be defined by (2.6). As N is a positive integer, it is known that x is an algebraic number [6]. Also, using the identity (2.7), and by Theorem 2.3.1 λ is an algebraic number.

Example 3.0.4

$$\textit{If}\ \ \ell = 4, \quad q = e^{-\pi\sqrt{N}}, \quad \textit{and} \quad N = 5,$$

$$\textit{then} \quad x = \frac{9}{64} - \frac{1}{16}\sqrt{5}, \tag{3.1}$$

$$\textit{and} \quad \lambda = \frac{1}{4} - \frac{1}{20}\sqrt{5}. \tag{3.2}$$

To determine x and λ in Maple, we use the command *identify* to give the corresponding algebraic numbers. Simply, if the input is a floating-point constant, then the *identify* command searches for an exact expression for the number. For example, if $x = 1.732050808$, using *identify* command determines that this is the constant $x = \sqrt{3}$.

```
> x := 1.732050808;
> identify(x);
                         (1/2)
                        3
```

For the *identify* command to succeed, enough digits must be provided to approximate the number.

For finding x in Example 3.0.4: we define the eta function by (2.5) and x for level 4 is given by (2.14). Then we convert the x as series of q into polynomial and substitute $q = -\pi\sqrt{5}$ into x. For this example, 46 digits were needed to determine x, at less than 46 the *identify* command does not give the quadratic irrational number (3.1).

In Maple, we write the code:

```
> e := proc (n) local i;
sum((-1)^i*q^((1/24)*(6*i-1)^2*n), i = -20 .. 20) end:
> x := convert(series
(e(1)^8*e(4)^8/(e(1)^8+16*e(4)^8)^2, q, 200), polynom);
> Digits := 46;
> evalf(subs(q = exp(-Pi*sqrt(5)), x));
      0.0008707514062631439744266457042952352849613525244
> identify(%);
                         9    1   (1/2)
                         -- - -- 5
                         64   16
```

Although there is a theoretical formula for λ, it's too complicated —indeed, impractical— to use in practice. Therefore, it is computed last, by summing the series. From (2.7), we can write λ as

$$\lambda = \frac{\left(\frac{\sqrt{\ell/N}}{2\pi\sqrt{1-4ax}}\right) - s_1}{s_2},$$

where

$$s_1 = \sum_{k=0}^{\infty} \binom{2k}{k} s(k)\, k\, x^k,$$

and

$$s_2 = \sum_{k=0}^{\infty} \binom{2k}{k} s(k)\, x^k.$$

For finding λ in Example 3.0.4, we only need 15 digits of x to determine λ. Using less than 15 digits, does not *identify* the quadratic irrational expression for λ in (3.2). The following code is used in Maple to compute λ:

```
> Digits := 15;
> l := 4;
> n := 5;
> x := 0.000870751406263149;
> s1 := 0;
> s2 := 0;
> k := 0;
> t := 1;
> while abs(t) > 10^(-15) do
s1 := evalf(s1+t*k):
s2 := evalf(s2+t):
t := t*x*(2*k+1)^3*2^3/(k+1)^3:
k := k+1:
do:
> k;
                              11
> evalf((sqrt(l/n)/(2*Pi*sqrt(1-64*x))-s1)/s2);
                     0.138196601125012
> identify(%);
                         1   1   (1/2)
                         - - -- 5
                         4   20
```

Numbers of digits and the terms in the series are modified depending on the speed of the convergence of the series. Some values of x and λ are easily computed using *identify* with a small number of digits, for instance, Example 3.0.4; we need 46 digits to identify x and 15 digits to identify λ. However some series need bigger numbers of digits to be identified. These series have much slower convergence.
For example: The series of $1/\pi$ of level 4 and $q = -e^{-\pi\sqrt{N}}$ that corresponds to $N = 7$ needed only 30 digits to identify x and λ. While the series of level 3 and $q = -e^{-2\pi\sqrt{N/3}}$ that corresponds to $N = 13$ needed 200 digits to identify x and λ.
In some cases the command *identify* does not work for finding the algebraic number of λ, and the command *PSLQ* can be used. Simply, when given a list (or a Vector) v

of n real numbers, the *PSLQ(v)* command outputs a list (or a vector) u of n integers such that $\sum_{i=1}^{n} u_i v_i$ is minimized. Thus the *PSLQ* function finds an integer relation between a vector of linearly dependent real numbers if the input has enough precision.

To find λ we take the numerical value of λ and include it in a vector with 1, and the same square root in x. This *PSLQ* command produces a minimal polynomial for λ that can be solved to give a quadratic irrational value of λ.

Example 3.0.5 *The series of* $1/\pi$ *of level 3 and* $q = e^{-2\pi\sqrt{N/3}}$ *that corresponds to* $N = 3$ *was found by using PSLQ.*

We used this code in Maple:

```
> Digits := 20:
> lambda := .14854314511050557756:
> with(IntegerRelations);
                  [LLL, LinearDependency, PSLQ]
> v := [lambda, 1, sqrt(3)]:
> u := PSLQ(v);
                          [22, -5, 1]
```

The *PSLQ* produces the vector u, and $\sum u_i \ v_i \quad \forall\ i = 1, 2, 3$ is minimized. If λ is a relation linear combination of 1 and $\sqrt{3}$, then since *PSLQ* produces the minimum, we have:

$22\lambda - 5 + \sqrt{3} = 0$, solving for λ gives $\lambda = \frac{5}{22} - \frac{1}{22}\sqrt{3}$.

Another way to determine the values of x and λ, is to use the modular equations of degree N that is satisfied by x. This method is illustrated in [6]. However, this method was only practical for small degrees of N.

Example 3.0.6 *The values of* x *and* λ *for the series of level 2 and degree* $N = 3$ *is found by using the modular equations*

Consider the level $\ell = 2$ and degree $N = 3$. Then $x = x(q)$ and $v = x(q^3)$ satisfy the modular equation:

$$x^4 + v^4 + 49152(x^4 v + x v^4) + 905969664(x^4 v^2 + x^2 v^4) + 7421703487488$$
$$(x^4 v^3 + x^3 v^4) + 22799473113563136 x^4 v^4 - 19332(x^3 v + x v^3) + 526860288$$
$$(x^3 v^2 + x^2 v^3) - 918519021568 x^3 v^3 + 312(x^2 v + x v^2) + 362266 x^2 v^2 - x v = 0.$$

For the value $q = e^{-2\pi\sqrt{3/2}}$, $\ell = 2$ and $N = 3$ we have:

$$v = x(q^3) = x\left(e^{-2\pi\sqrt{3/2}}\right) = x\left(e^{-2\pi/\sqrt{6}}\right) = x,$$

and the modular equation simplifies and factorizes to:

$$x^2(144x+1)(2304x-1)(1024x+1)^2(-1+256x)^2 = 0. \tag{3.3}$$

We deduce that:

$$x \in \{0, \frac{-1}{44}, \frac{1}{2304}, \frac{-1}{1024}, \frac{1}{256}\}.$$

By using numerical approximation to x to determine which root of (3.3) to select, we have:

$$x\left(e^{-2\pi\sqrt{3/2}}\right) = x\left(e^{-2\pi/\sqrt{6}}\right) = \frac{1}{2304}.$$

Using Theorem 2.3.2 we have:

$$\begin{aligned} M(q) =& \frac{Z(q)}{Z(q^3)} = \frac{u(q^3)}{u(q)} \frac{q\frac{d}{dq}\log x(q)}{q\frac{d}{dq}\log x(q^3)}, \\ =& 3\frac{\sqrt{1-256v}}{\sqrt{1-256x}} \frac{v}{x} \frac{dx}{dv}. \end{aligned}$$

The derivative dx/dv can be calculated by differentiating the modular equation implicitly and therefore $M(q)$ becomes an algebraic function of x and v. Differentiating $M(q)$ with respect to x and substituting q gives:

$$\left.\frac{dM}{dx}\right|_{q=(-2\pi/sqrt6)} = 1728.$$

Using the formula for λ from Theorem 2.3.2 gives:

$$\lambda = \frac{1/2304}{6} \times 1728 = \frac{1}{8}.$$

3.1 Results

In this section we first present tables of all the values of x and λ for series of $1/\pi$. A total of 90 series are included, 76 of which are believed to be new. In the discussion section, we analyze these results and observe the important points and patterns.

Tables

This section contains tables of values of the parameters x and λ. Tables 3.1–3.10 with values of x and λ that give series for $1/\pi$ are organized according to the level $\ell \in \{1, 2, 3, 4\}$ and according to $q > 0$ or $q < 0$. Entries within each table are organized according to the degree N. An asterisk "*" next to the degree indicates that the series was already given in [4]; there are 14 of these series and they belong to the levels $\ell \in \{1, 2\}$. The other 76 series are believed to be new.

q	N	x	λ
$e^{-2\pi\sqrt{N}}$	6^*	$\frac{1399}{8489664} - \frac{247}{2122416}\sqrt{2}$	$\frac{25}{276} - \frac{5}{276}\sqrt{2}$
	8	$\frac{209}{97336} - \frac{18473}{12167000}\sqrt{2}$	$\frac{78}{713} - \frac{375}{9982}\sqrt{2}$
	10^*	$\frac{4927}{210720960} - \frac{51}{4077880}\sqrt{5}$	$\frac{83}{1116} - \frac{1}{93}\sqrt{5}$
	12	$\frac{389}{1706850} - \frac{20213}{161716500}\sqrt{3}$	$\frac{2938}{35673} - \frac{250}{11891}\sqrt{3}$
	13	$-\frac{3193}{525614400} + \frac{41}{24334000}\sqrt{13}$	$\frac{103}{1548} - \frac{125}{20124}\sqrt{13}$
	15	$\frac{10968319}{90769370526} - \frac{122629507}{2269234263150}\sqrt{5}$	$\frac{1052}{13629} - \frac{20}{1239}\sqrt{5}$
	16	$-\frac{190338695}{34106789907} - \frac{19939227}{5052857764}\sqrt{2}$	$\frac{10}{99} - \frac{10}{231}\sqrt{2}$
	22^*	$\frac{2914279}{14627903313600} - \frac{95403}{677217746000}\sqrt{2}$	$\frac{1013}{19908} - \frac{125}{10428}\sqrt{2}$
	28	$\frac{8145698488}{79267303539275} - \frac{570145303}{14679130284125}\sqrt{7}$	$\frac{90352}{1274007} - \frac{59000}{3822021}\sqrt{7}$
	37	$-\frac{9180598102}{365659375737799440000} + \frac{27940649}{67714699210703600}\sqrt{37}$	$\frac{144743}{3687948} - \frac{2054375}{955178532}\sqrt{37}$
	58^*	$\frac{1399837865393267}{6884754959794970900700000} - \frac{1203441508269}{318738655546063467625000}\sqrt{29}$	$\frac{6117973}{195168708} - \frac{4314395}{2223529209}\sqrt{29}$

Table 3.1: Irrational Series of level $l = 1$ for $q = e^{-2\pi\sqrt{N}}$

q	N	x	λ
$-e^{-\pi\sqrt{N}}$	35	$\frac{9}{20480}-\frac{161}{819200}\sqrt{5}$	$\frac{873}{8246}-\frac{96}{4123}\sqrt{5}$
	75	$-\frac{36983}{588791808}+\frac{18377}{654213120}\sqrt{5}$	$\frac{6965}{92158}-\frac{800}{46079}\sqrt{5}$
	91	$\frac{2927165}{2173353984}-\frac{60137}{160989184}\sqrt{13}$	$\frac{15}{154}-\frac{160}{9009}\sqrt{13}$
	99	$\frac{9487199}{28334096384}-\frac{18166603}{311675060224}\sqrt{33}$	$\frac{2085}{22078}-\frac{120}{11039}\sqrt{33}$
	235*	$-\frac{2796157855}{81160240398336}+\frac{578925837}{37574185369600}\sqrt{5}$	$\frac{1425191}{22391082}-\frac{214720}{11195541}\sqrt{5}$
	267*	$-\frac{1423834769537}{76885078511663972352}+\frac{18865772964857}{961063481395799654400}\sqrt{89}$	$\frac{2012743871}{51659265306}-\frac{33230537000}{16091861142819}\sqrt{89}$
	427*	$-\frac{265143898070738S1}{526643727429777408000}+\frac{251468129201653}{390106464762798080000}\sqrt{61}$	$\frac{1594143977}{25145035062}-\frac{14138201440}{2300770708173}\sqrt{61}$

Table 3.2: Irrational Series of level $l=1$ for $q=-e^{-\pi\sqrt{N}}$

q	N	x	λ
$e^{-2\pi\sqrt{N/2}}$	4	$-\frac{457}{19208}+\frac{325}{19208}\sqrt{2}$	$\frac{6}{35}-\frac{3}{70}\sqrt{2}$
	6	$\frac{373}{263538}-\frac{425}{527076}\sqrt{3}$	$\frac{14}{115}-\frac{2}{115}\sqrt{3}$
	7	$\frac{249}{12544}-\frac{11}{784}\sqrt{2}$	$\frac{9}{56}-\frac{3}{56}\sqrt{2}$
	8	$-\frac{221}{194481}+\frac{209}{259308}\sqrt{2}$	$\frac{626}{4991}-\frac{162}{4991}\sqrt{2}$
	14	$\frac{102376}{855036681}-\frac{38675}{855036081}\sqrt{7}$	$\frac{4528}{50995}-\frac{552}{50995}\sqrt{7}$
	15	$\frac{6449}{5531904}+\frac{95}{115248}\sqrt{2}$	$\frac{29}{280}-\frac{9}{280}\sqrt{2}$
	17	$\frac{2177}{20736}-\frac{11}{432}\sqrt{17}$	$\frac{31}{208}-\frac{81}{3536}\sqrt{17}$
	21^*	$\frac{150889}{300259584}-\frac{1925}{9383112}\sqrt{6}$	$\frac{13}{140}-\frac{1}{56}\sqrt{6}$

Table 3.3: Irrational Series of level $l = 2$ for $q = e^{-2\pi\sqrt{N/2}}$

q	N	x	λ
$e^{-2\pi\sqrt{N/2}}$	35^*	$\frac{18287}{11757312} - \frac{1265}{2571912}\sqrt{10}$	$\frac{1543}{16120} - \frac{147}{8060}\sqrt{10}$
	39^*	$\frac{243407089}{5108829513984} - \frac{3585725}{106433948208}\sqrt{2}$	$\frac{5983}{83720} - \frac{2097}{83720}\sqrt{2}$
	41	$\frac{54600721}{5802782976} - \frac{88825}{60445656}\sqrt{41}$	$\frac{79}{592} - \frac{1863}{121360}\sqrt{41}$
	51	$-\frac{13062107489}{1026527766038784} - \frac{33000275}{10692997562904}\sqrt{17}$	$\frac{10735}{171080} - \frac{2763}{363545}\sqrt{17}$
	65^*	$\frac{86801836241}{242945341583616} + \frac{112150445}{2530680641496}\sqrt{65}$	$\frac{9887}{123830} - \frac{5184}{804895}\sqrt{65}$
	95^*	$\frac{427925331521}{5029625975060736} + \frac{6303935495}{104783874480432}\sqrt{2}$	$\frac{50572309}{741263320} + \frac{23655969}{741263320}\sqrt{2}$

Table 3.4: Irrational Series of level $l = 2$ for $q = e^{-2\pi\sqrt{N/2}}$

q	N	x	λ
$-e^{-\pi\sqrt{N}}$	17	$\frac{103}{32768} - \frac{25}{32768}\sqrt{17}$	$\frac{3}{20} - \frac{3}{170}\sqrt{17}$
	21	$\frac{-97}{9216} + \frac{7}{1152}\sqrt{3}$	$\frac{53}{364} - \frac{4}{91}\sqrt{3}$
	33	$-\frac{1867}{1179648} + \frac{325}{1179648}\sqrt{33}$	$\frac{73}{620} - \frac{37}{3410}\sqrt{33}$
	49	$\frac{85}{36864} - \frac{253}{290304}\sqrt{7}$	$\frac{773}{5828} - \frac{48}{1457}\sqrt{7}$
	57	$-\frac{542267}{2832334848} + \frac{71825}{2832334848}\sqrt{57}$	$\frac{157}{1820} - \frac{101}{17290}\sqrt{57}$
	73	$\frac{152107}{4718592} - \frac{160225}{42467328}\sqrt{73}$	$\frac{4511}{32660} - \frac{14073}{1192090}\sqrt{73}$
	85	$-\frac{22861961}{10809345024} + \frac{231035}{450389376}\sqrt{17}$	$\frac{1531}{13780} - \frac{7614}{409955}\sqrt{17}$
	177*	$-\frac{38480821035067}{86466866929622581248} + \frac{964131876175}{28822288976540860416}\sqrt{177}$	$\frac{3562829}{74444860} - \frac{3950637}{2196123370}\sqrt{177}$
	253*	$-\frac{12633605109401}{680849241375744} + \frac{158715635975}{28368718390656}\sqrt{11}$	$\frac{7152017}{57366140} - \frac{995652}{31551377}\sqrt{11}$

Table 3.5: Irrational Series of level $l = 2$ for $q = -e^{-\pi\sqrt{N}}$

q	N	x	λ
$e^{-2\pi\sqrt{N/3}}$	3	$-\frac{1}{6}+\frac{7}{72}\sqrt{3}$	$\frac{5}{22}-\frac{1}{22}\sqrt{3}$
	6	$\frac{463}{125000}-\frac{91}{62500}\sqrt{6}$	$\frac{33}{230}-\frac{3}{230}\sqrt{6}$
	7	$-\frac{17}{2916}+\frac{13}{5832}\sqrt{7}$	$\frac{1}{6}-\frac{1}{42}\sqrt{7}$
	8	$-\frac{265}{108}+\frac{17}{12}\sqrt{3}$	$\frac{29}{138}-\frac{3}{46}\sqrt{3}$
	10	$\frac{223}{157464}-\frac{35}{78732}\sqrt{10}$	$\frac{11}{90}-\frac{1}{90}\sqrt{10}$
	11	$-\frac{97}{71879}+\frac{25}{31944}\sqrt{3}$	$\frac{43}{330}-\frac{3}{110}\sqrt{3}$
	13	$\frac{17743}{39366}-\frac{4921}{39366}\sqrt{13}$	$\frac{10}{51}-\frac{22}{663}\sqrt{13}$
	19	$-\frac{4261}{22781250}+\frac{3913}{91125000}\sqrt{19}$	$\frac{49}{510}-\frac{73}{9690}\sqrt{19}$
	20	$\frac{205694}{47832147}-\frac{13195}{5314683}\sqrt{3}$	$\frac{128}{1065}-\frac{12}{355}\sqrt{3}$
	31	$-\frac{684197}{33215062500}+\frac{245791}{66430125000}\sqrt{31}$	$\frac{1007}{13530}-\frac{1877}{419430}\sqrt{31}$
	34	$\frac{3555313}{278957081304}-\frac{152425}{69739270326}\sqrt{34}$	$\frac{2033}{28710}-\frac{197}{48807}\sqrt{34}$
	59	$-\frac{730612447}{1641786993734274}+\frac{187475575}{729683108326344}\sqrt{3}$	$\frac{5472653}{101652870}-\frac{351123}{33884290}\sqrt{3}$

Table 3.6: Irrational Series of level $l=3$ for $q=e^{-2\pi\sqrt{N/3}}$

q	N	x	λ
	2	$-\frac{265}{108}-\frac{17}{12}\sqrt{3}$	the series diverges
	3	$-\frac{1}{6}-\frac{7}{72}\sqrt{3}$	the series diverges
	4	$-\frac{18551}{421875}-\frac{561}{31250}\sqrt{6}$	the series diverges
	7	$-\frac{17}{2916}-\frac{13}{5832}\sqrt{7}$	the series diverges
	11	$-\frac{97}{71874}-\frac{25}{31944}\sqrt{3}$	$\frac{43}{330}+\frac{3}{110}\sqrt{3}$
$-e^{-\pi\sqrt{N/3}}$	13	$\frac{23}{1458}-\frac{7}{1458}\sqrt{13}$	$\frac{2}{9}-\frac{2}{117}\sqrt{13}$
	19	$-\frac{4261}{22781250}-\frac{3913}{91125000}\sqrt{19}$	$\frac{49}{510}+\frac{73}{9690}\sqrt{19}$
	31	$-\frac{684197}{33215062500}-\frac{245791}{66430125000}\sqrt{31}$	$\frac{1007}{13530}+\frac{1877}{419430}\sqrt{31}$
	33	$-\frac{523}{8192}+\frac{91}{8192}\sqrt{33}$	$\frac{96}{493}-\frac{93}{5423}\sqrt{33}$
	59	$-\frac{730612447}{1641786993734274}-\frac{187475575}{729683108326344}\sqrt{3}$	$\frac{5472653}{101652870}+\frac{351123}{33884290}\sqrt{3}$
	65	$-\frac{649}{1728}+\frac{5}{48}\sqrt{13}$	$\frac{83}{435}-\frac{64}{1885}\sqrt{13}$

Table 3.7: Irrational Series of level $l=3$ for $q=-e^{-\pi\sqrt{N/3}}$

q	N	x	λ
$-e^{-\pi\sqrt{N/3}}$	73	$\frac{1555}{46656} - \frac{91}{23328}\sqrt{73}$	$\frac{73}{207} - \frac{202}{15111}\sqrt{73}$
	97	$\frac{33161}{5971968} - \frac{3367}{5971968}\sqrt{97}$	$\frac{16}{99} - \frac{103}{9603}\sqrt{97}$
	121	$\frac{24323}{221184} - \frac{1725}{90112}\sqrt{33}$	$\frac{8}{45} - \frac{1}{45}\sqrt{33}$
	145	$-\frac{1769693}{161243136} + \frac{146965}{161243136}\sqrt{145}$	$\frac{314}{2115} - \frac{523}{62335}\sqrt{145}$
	169	$\frac{14339}{1259712} - \frac{12925}{4094064}\sqrt{13}$	$\frac{941}{5865} - \frac{64}{1955}\sqrt{13}$
	185	$-\frac{14983669}{3061257408} + \frac{68425}{85034928}\sqrt{37}$	$\frac{12079}{91635} - \frac{3392}{226033}\sqrt{37}$
	241	$\frac{489978007}{729000000} - \frac{15781129}{364500000}\sqrt{241}$	$\frac{22051}{124785} - \frac{273526}{30073185}\sqrt{241}$
	265	$-\frac{30814033}{20155392000} - \frac{1892891}{20155392000}\sqrt{265}$	$\frac{53662}{498249} - \frac{119771}{26407197}\sqrt{265}$
	409	$\frac{77088425801}{5971968000000} - \frac{3811777333}{5971968000000}\sqrt{409}$	$\frac{1094279}{7115265} - \frac{18207572}{2910143385}\sqrt{409}$

Table 3.8: Irrational Series of level $l = 3$ for $q = -e^{-\pi\sqrt{N/3}}$

q	N	x	λ
$e^{-\pi\sqrt{N}}$	2	$-\frac{7}{8}+\frac{5}{8}\sqrt{2}$	$\frac{2}{7}-\frac{1}{14}\sqrt{2}$
	4	$-35+\frac{99}{4}\sqrt{2}$	$\frac{2}{7}-\frac{2}{21}\sqrt{2}$
	5	$\frac{9}{64}-\frac{1}{16}\sqrt{5}$	$\frac{1}{4}-\frac{1}{20}\sqrt{5}$
	9	$\frac{97}{64}-\frac{7}{8}\sqrt{3}$	$\frac{1}{4}-\frac{1}{12}\sqrt{3}$
	13	$\frac{649}{64}-\frac{45}{16}\sqrt{13}$	$\frac{1}{4}-\frac{7}{156}\sqrt{13}$
	15	$\frac{47}{8192}-\frac{21}{8192}\sqrt{5}$	$\frac{3}{22}-\frac{4}{165}\sqrt{5}$
	25	$\frac{51841}{64}-\frac{1449}{4}\sqrt{5}$	$\frac{1}{4}-\frac{1}{12}\sqrt{5}$
	37	$\frac{1555849}{64}-\frac{63945}{16}\sqrt{37}$	$\frac{1}{4}-\frac{101}{3108}\sqrt{37}$

Table 3.9: Irrational Series of level $l=4$ for $q=e^{-\pi\sqrt{N}}$

q	N	x	λ
	6	$-\frac{17}{64}+\frac{3}{16}\sqrt{2}$	$\frac{1}{4}-\frac{1}{12}\sqrt{2}$
	7	$-2024+765\sqrt{7}$	$\frac{16}{57}-\frac{8}{133}\sqrt{7}$
	10	$-\frac{161}{64}+\frac{9}{8}\sqrt{5}$	$\frac{1}{4}-\frac{1}{15}\sqrt{5}$
	12	$-\frac{13}{512}+\frac{15}{1024}\sqrt{3}$	$\frac{13}{66}-\frac{2}{33}\sqrt{3}$
$-e^{-\pi\sqrt{N}}$	16	$\frac{35}{512}-\frac{99}{2048}\sqrt{2}$	$\frac{3}{14}-\frac{2}{21}\sqrt{2}$
	18	$-\frac{4801}{64}+\frac{245}{8}\sqrt{6}$	$\frac{1}{4}-\frac{1}{14}\sqrt{6}$
	22	$-\frac{19601}{64}+\frac{3465}{16}\sqrt{2}$	$\frac{1}{4}-\frac{17}{132}\sqrt{2}$
	28	$-\frac{253}{512}+\frac{765}{4096}\sqrt{7}$	$\frac{25}{114}-\frac{8}{133}\sqrt{7}$
	58	$-\frac{192119201}{64}+\frac{4459455}{8}\sqrt{29}$	$\frac{1}{4}-\frac{37}{957}\sqrt{29}$

Table 3.10: Irrational Series of level $l=4$ for $q=-e^{-\pi\sqrt{N}}$

3.2 Discussion

In Tables 3.1-3.10 the values of x and λ were presented to give a total of 90 series for $1/\pi$. These series were obtained from the identity (2.7). It is obvious that the values of x and λ in the quadratic irrational forms share the same square roots for each degree. In most cases, it was a square root of the level, the degree or a divisor of the degree. There are 4 examples where the conjugate series (i.e., the series obtained by replacing each quadratic irrational with its conjugate) also gives a series for $1/\pi$, but the convergence is much slower. These series are from level 3 and correspond to the degrees $N = \{11, 19, 31, 59\}$, where these values of N satisfy the condition $N \equiv 3 \pmod 4$. As for the convergence, all series found are convergent except for 4 series. These series are from level 3 and correspond to $N = \{2, 3, 4, 7\}$. They diverge because $x > \frac{1}{108}$; and $\frac{1}{108}$ is the radius of convergence for $\ell = 3$.

3.3 Conclusion

We have listed 90 Ramanujan-type series for $1/\pi$ that involve quadratic irrationals of the levels $\ell \in \{1, 2, 3, 4\}$. Of the 90 series, 76 are believed to be new and 14 were previously known. There are series of higher levels; $\ell \in \{5, 6, 7, 8, 9, 10\}$. The main difference is that, instead of the 2-term recurrence relations in Theorem 2.3.1, the underlying sequences satisfy three-term relation. Heun functions appears instead of hypergeometric functions. Only six examples are known, and these are for levels 5,6 (3 cases), 8, and 9. For more details on rational series for $1/\pi$ of the levels $\ell \in \{5, 6, 8, 9\}$ see [6]. In addition, there are series of level $\ell = 7, 10, 18$, not investigated in this thesis but details are in [12] and [13]. It is not known if there other series for $1/\pi$ that can be obtained from the identity (2.7). Future work could involve rational and irrational series for $1/\pi$ of other levels.

Bibliography

[1] N. D. Baruah, B. C. Berndt, and H. H. Chan, Ramanujan's series for $1/\pi$: A survey, *Amer. Math. Monthly,* **116** (2009) 567–587.

[2] B. C. Berndt, and H. H. Chan, Eisenstein series and approximations to π, *Illinois J. Math,* **45** (2001) 75–90.

[3] J. M. Borwein, and P. B. Borwein, Pi and the AGM – A study in analytic number theory and computational complexity, Wiley, New York, NY (1987).

[4] J. M. Borwein, and P. B. Borwein, *More Ramanujan-type series for* $1/\pi$*,* In: Ramanujan revisited (Urbana-Champaign, Ill., 1987), 359–374, Boston: Academic Press, 1988.

[5] H. H. Chan, S. H. Chan, and Z.-G. Liu, Domb's numbers and Ramanujan-Sato type series for $1/\pi$, *Adv. Math.*, **186** (2004) 396–410.

[6] H. H. Chan, and S. Cooper, Rational Analogues of Ramanujan's series for $1/\pi$, *Math. Proc. Cambridge Philos. Soc.,* (published online 17 May 2012) . DOI: 10.1017/S0305004112000254.

[7] H. H. Chan, W. C. Liaw, and V. Tan, Ramanujan's class invariant λ_n and a new class of series for $1/\pi$, *J. London Math. Soc.*, **64** (2001) 93–106.

[8] S. Chowla, Series for $1/K$ and $1/K^2$, *J. London Mathematical Society,* **3** (1928) 9–12.

[9] D. Chudnovsky, and G. Chudnovsky, *Approximation and complex multiplication according to Ramanujan,* In: Ramanujan revisited (Urbana-Champaign, Ill., 1987), 375 – 472, Boston: Academic Press, 1988.

[10] S. Cooper, Inversion formulas for elliptic functions, *Proc. London Math. Soc.,* **99** (2009) 461–483.

[11] S. Cooper, Series and iterations for $1/\pi$, *Acta Arith,* **141** (2010) 33–58.

[12] S. Cooper, Level 10 analogues of Ramanujan's series $1/\pi$, *J. Ramanujan Math. Soc,* **27** (2012) 75–92.

[13] S. Cooper, Sporadic sequences, modular forms and new series for $1/\pi$, *Ramanujan J,* (published online 26 April 2012) DOI: 10.1007/s11139-011-9357-3.

[14] J. Guillera, History of the formulas and algorithms for π, *Contemp. Math. Gems in Experimental Mathematics* **517** (2010) 173–188.

[15] S. Ramanujan, Modular equations and approximations to π, *Quart. J. Math,* **45** (1914) 350–372.

[16] M. Rogers, New ${}_5F_4$ hypergeometric transformation, three-variable Mahler measures, and formulas for $1/\pi$, *Ramanujan J,* **18** (2009) 327–340.

[17] T. Sato, Apéry numbers and Ramanujan's series for $1/\pi$, *Abstract of a talk presented at the Annual meeting of the Mathematical Society of Japan* (2002) 28–31.