CALCULATION OF FUNDAMENTAL UNITS IN SOME TYPES OF QUARTIC NUMBER FIELDS

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ABSTRACT

Dirichlet's theorem describing the structure of the unit group of the ring of integers of an algebraic number field shows that the units are generated by a primitive root of unity of the field plus a finite set of units called a fundamental system of units. However Dirichlet's theorem does not suggest any method by which a fundamental system of units can be obtained. In this thesis we consider the problem of calculating a fundamental system of units for certain types of quartic field which are a quadratic extension of a quadratic field $Q(\delta)$. Our attention is mainly centered on type I quartic fields, that is quartic fields for which $Q(\delta)$ is complex. In such cases a fundamental system of units contains a single unit called a fundamental unit.

To calculate fundamental units of type I quartic fields we use the simple continued fraction algorithm, real quadratic field case as a guide. This topic is reviewed in chapter one where we also note Voronoi's view of simple continued fractions in terms of relative minima of a Z module.

In chapter two we consider the idea of relative minima of a module over a ring of complex quadratic integers. Basically we generalize the simple continued fraction algorithm which calculates best approximations to a real number using rational integer coefficients to an algorithm which calculates best approximations to a complex number using complex quadratic integer coefficients. The ideas are developed with respect to an arbitrary complex quadratic field $Q(\delta)$ and show many similarities to the simple continued fraction algorithm. (Existing work of this nature restricts its attention to cases where $Q(\delta)$ has class number one). We obtain an algorithm which is periodic for complex numbers w satisfying

$w^2 \in Q(\delta)$, $w \notin Q(\delta)$. This enables us to calculate units of type I quartic fields.

In chapter three we consider quartic fields $Q(\Gamma)$ which are a quadratic extension of a quadratic field $Q(\delta)$. In section one we express the ring of integers of $Q(\Gamma)$ in terms of the integers of $Q(\delta)$ thereby recognising four forms which these rings may take. In section two we consider the problem of calculating fundamental units of type I quartic fields. The algorithm developed in chapter two is only guaranteed to locate a fundamental unit when the ring of integers of $Q(\Gamma)$ is of the simplest of the four forms mentioned above. A modified version of the algorithm allows us to calculate a fundamental unit when the ring of integers of $Q(\Gamma)$ is of the second simplest form. For the two remaining forms we obtain a unit U which may or may not be fundamental. We therefore develop an algorithm which calculates a fundamental unit from U. To illustrate the use of our algorithms we calculate fundamental units for the type I quartic fields

$$Q(\sqrt[4]{D}),\ D \in Z,\ -99 \le D \le -1$$

Finally in section three we consider the calculation of a fundamental system of units for type IIb quartic fields, that is semi-real quartic fields which are a quadratic extension of a real quadratic field. A connection between type IIb and type I quartic fields enables us to calculate fundamental systems of units for type IIb quartic fields.

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CONTENTS

REFERENCE PAGES FOR NOTATION AND TERMINOLOGY

The list which follows is not exhaustive. We have omitted most standard notation and terminology plus most notation and terminology which has a localised usage.