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CALCULATION OF FUNDAMENTAL UNITS
IN SOME TYPES OF
QUARTIC NUMBER FIELDS

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ABSTRACT

Dirichlet's theorem describing the structure of the unit group of the ring of integers of an algebraic number field shows that the units are generated by a primitive root of unity of the field plus a finite set of units called a fundamental system of units. However Dirichlet's theorem does not suggest any method by which a fundamental system of units can be obtained. In this thesis we consider the problem of calculating a fundamental system of units for certain types of quartic field which are a quadratic extension of a quadratic field $Q(\delta)$. Our attention is mainly centered on type I quartic fields, that is quartic fields for which $Q(\delta)$ is complex. In such cases a fundamental system of units contains a single unit called a fundamental unit.

To calculate fundamental units of type I quartic fields we use the simple continued fraction algorithm, real quadratic field case as a guide. This topic is reviewed in chapter one where we also note Voronoi's view of simple continued fractions in terms of relative minima of a Z module.

In chapter two we consider the idea of relative minima of a module over a ring of complex quadratic integers. Basically we generalize the simple continued fraction algorithm which calculates best approximations to a real number using rational integer coefficients to an algorithm which calculates best approximations to a complex number using complex quadratic integer coefficients. The ideas are developed with respect to an arbitrary complex quadratic field $Q(\delta)$ and show many similarities to the simple continued fraction algorithm. (Existing work of this nature restricts its attention to cases where $Q(\delta)$ has class number one). We obtain an algorithm which is periodic for complex numbers w satisfying

$w^2 \in Q(\delta)$, $w \notin Q(\delta)$. This enables us to calculate units of type I quartic fields.

In chapter three we consider quartic fields $Q(\Gamma)$ which are a quadratic extension of a quadratic field $Q(\delta)$. In section one we express the ring of integers of $Q(\Gamma)$ in terms of the integers of $Q(\delta)$ thereby recognising four forms which these rings may take. In section two we consider the problem of calculating fundamental units of type I quartic fields. The algorithm developed in chapter two is only guaranteed to locate a fundamental unit when the ring of integers of $Q(\Gamma)$ is of the simplest of the four forms mentioned above. A modified version of the algorithm allows us to calculate a fundamental unit when the ring of integers of $Q(\Gamma)$ is of the second simplest form. For the two remaining forms we obtain a unit U which may or may not be fundamental. We therefore develop an algorithm which calculates a fundamental unit from U . To illustrate the use of our algorithms we calculate fundamental units for the type I quartic fields

$$Q(\sqrt[4]{D}), D \in \mathbb{Z}, -99 \leq D \leq -1$$

Finally in section three we consider the calculation of a fundamental system of units for type IIb quartic fields, that is semi-real quartic fields which are a quadratic extension of a real quadratic field. A connection between type IIb and type I quartic fields enables us to calculate fundamental systems of units for type IIb quartic fields.

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REFERENCE PAGES FOR NOTATION AND TERMINOLOGY

The list which follows is not exhaustive. We have omitted most standard notation and terminology plus most notation and terminology which has a localised usage.

$Q(\alpha), Z(\alpha), Q(\alpha)(\beta), Z(\alpha)(\beta)$	3	$M_k(w), E_{k,j}, R_{k,j}, R_k$	49
$[Q(\alpha):Q], N, N_\alpha$	3	basic, non-basic	52
$R[A_1, \dots, A_k]$ where $R = Z, Z(\alpha)$	5	$I_k, g_k, \sigma_k = (a_k + \delta)/c$	52
packing constant	5	$\kappa_k, \lambda_k, B_k, W_k$	53
$d, \delta, Q(\delta), Z(\delta)$	6	standard representation of $M_k(w)$	53
c	6	$(\alpha, \beta) W_k$ allowable	53
ω	7	ψ_k	53
Δ	7	$w_{k,1}, w_{k,2}, \eta_{k,1}, \eta_{k,2}$	57
α'	7	$\eta_{k,3}, \eta_{k,4}, S_k$	66
$\varepsilon(d)$	8	periodic relative minima, period, length, q	89
$\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$	8	$\sqrt{\gamma/h}$	91
I'	8	$N_\delta(R)$ for $R \in M(w)/A$	93
(Δ/p)	8	θ_k	94
$h(d)$	9	unit of $M(w)$	96
$\{x\}$	11	minimal period	99
$Z(\delta)^+$	12	$M^*(w), M_k^*(w)$	99
$A^*(\text{reverse}), A , \bar{A}$	26	symmetric minimal period	101
$M(w), W$	26	$Z(\delta)CF$	125
$M(w)/X, (M(w)/X)/Y$	27	$Q(\Gamma)$	133
relative minimum	28	non-square and rational square-free	133
A_0	31	$\gamma = a_1 a_2^p$	134
\sim , equivalent relative minima	32	$\gamma_1, \gamma_2, I(\gamma_1, \gamma_2), A^*$ (conjugate) form 1, 2, 3, 4	136
E_k, A_k	32	type I, II quartic fields	151
$A_k^{(j)}, \alpha_k, \beta_k, \alpha_k^{(j)}, \beta_k^{(j)}$	33		
chain, complete chain, half chain	33		

type IIa, IIb, IIc quartic	$T_f = (\mu_1 + \mu_2 W_m) / g_m, b(\mu_2)$	162
fields	$L(\sqrt{\gamma})$	153 174
U_f	$D = rs^2 t^3$	157 183
V_f	$A^*, A', A'^*, A^{*'} $	160 193